Perspectives of discontinuous Galerkin method for the numerical solution of computational fluid dynamics problems

Vít Dolejší

Charles University Prague
Faculty of Mathematics and Physics

Programy a algoritmy numerické matematiky 17
8. - 13. června 2014
Dolní Maxov

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Computational fluid dynamics (CFD)

- WIKI: CFD uses numerical methods to solve and analyze problems involving fluid flows,
- turbomachinery, aeroacoustic, aeronautics, car industry, biomedical, electronics, environmental protection,
- problems are described by partial differential equations (PDEs).

The main goal

compute the quantities of interest
under the given accuracy
within the shortest possible time.

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How to achieve the main goal

- suitable discretization:
  numerical method & domain partition $\mathscr{T}_h$, τ_k,
- efficient solution of arising algebraic systems
- accurate and efficient error estimates.

Requirements

- numerical method: accurate, robust, parallelizable
- algebraic solver: iterative, stopping criteria
- a posterior error analysis: accurate and fast approximation of error, algebraic error
- mesh adaptation: h, hp variants, anisotropy
- time step adaptation: explicit/implicit, order adaptation

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Google searching

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- “CFD software finite volume method” – 161 000 links
- “CFD software spectral method” – 168 000 links
- “CFD software discontinuous Galerkin method” – 25 000 links

www.cfd-online.com – list of CFD codes (public and commercial)

- finite element methods (FEM) – 50%
- finite volume methods (FVM) – 40%
- spectral methods (SM) – 10%
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FEM – piecewise polynomial continuous approximation

- high order of accuracy supported by theory
- efficient for problems with continuous solutions

FVM – piecewise constant discontinuous approximation

- low order of accuracy (commercial codes at most 2^{nd} order)
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discontinuous Galerkin method

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- a little complicated formulation and theoretical analysis
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1D convection-diffusion problem

$$-\varepsilon u'' + u' = 1 \quad \text{in } \Omega := (0,1), \qquad u(0) = u(1) = 0,$$

$0 < \varepsilon \ll 1$ convection dominanting problem

$\Rightarrow$ boundary layer at $x = 1$ proportional to ε.

Numerical tests

- FEM with continuous P_1 approximation with $h = 1/32$,
- FVM piecewise constant approximation with $h = 1/32$,
- DGM using P_4 approximation with $h = 1/8$.

$\Rightarrow$ 32 degrees of freedom for all cases.

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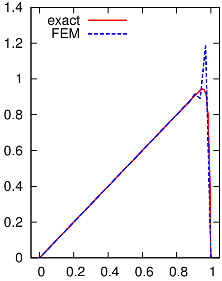
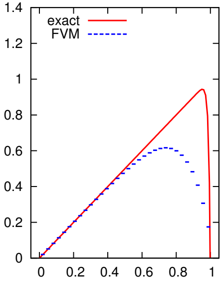
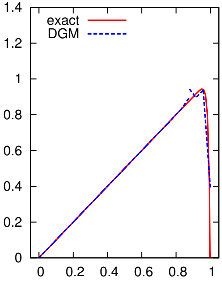


FEM

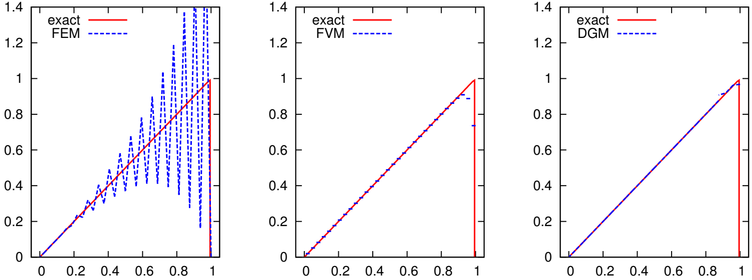


FVM



DGM

Comparison of FVM, FEM and DGM with $\varepsilon = 10^{-3}$



FEM FVM DGM

Why higher order approximation?

the aim is $\|u - u_h\| \leq \omega$ & DOF minimal

Theoretical as well as experimental results

$\|u - u_h\| \approx C h^{p+1}$, $h \approx (\#\mathscr{T}_h)^{1/d}$, $d = 2, 3$
$p =$ polyn. degree, $\mathrm{DOF} \approx \frac{1}{d!}(p+1)\dots(p+d)\#\mathscr{T}_h$

number of DOF necessary to fulfil $\|u - u_h\| \leq \omega$:

d	ω	P_1	P_2	P_3	P_4	P_5	P_6
2	1.00E-02	301	130	101	95	98	105
	1.00E-03	3001	601	317	238	211	202
	1.00E-04	30001	2785	1001	598	453	390
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	1.00E-03	94869	6001	1779	947	665	541
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d	ω	P_1	P_2	P_3	P_4	P_5	P_6
2	1.00E-02	301	130	101	95	98	105
	1.00E-03	3001	601	317	238	211	202
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Why higher order approximation?

the aim is $\|u - u_h\| \leq \omega$ & DOF minimal

Theoretical as well as experimental results

$\|u - u_h\| \approx C h^{p+1}$, $h \approx (\#\mathscr{T}_h)^{1/d}$, $d = 2, 3$

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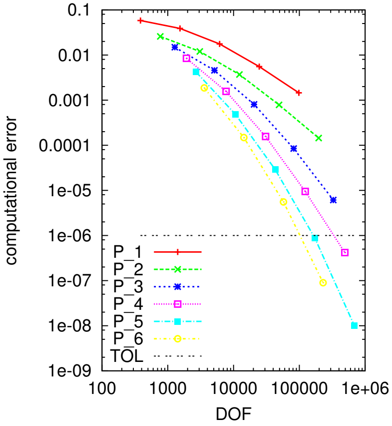
Practical demonstration of the efficiency

- scalar linear convection-diffusion equation solved by DGM



1. Introduction

2. DGM for problems of CFD

3. Solution strategies

4. Mesh adaptation

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System of convection-diffusion equations

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \mathbf{f}_s(\mathbf{w}) = \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \mathbf{R}_s(\mathbf{w}, \nabla \mathbf{w}), \qquad (1)$$

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- closing relations, e.g., from thermodynamic, BC + IC.

compressible Navier-Stokes equations, shallow water equations, etc.

Space of piecewise polynomial functions

$$S_h^p := \{v;\ v \in L^2(\Omega),\ v|_K \in P_p(K)\ \forall K \in \mathscr{T}_h\},\ \mathbf{S}_{hp} := (S_h^p)^N.$$

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Space-time discretization, $w(t_k) \approx w_h^k \in \mathbf{S}_{hp}$

$$\frac{1}{\tau_k}\left(\mathbf{w}_h^k - \mathbf{w}_h^{k-1}, \varphi_h\right) + \mathbf{c}_h(\mathbf{w}_h^k, \varphi_h) = 0 \quad \forall \varphi_h \in \mathbf{S}_{hp}, \ k = 1, 2, \dots$$

$$\begin{aligned}
\mathbf{c}_h(\mathbf{w}_h, \varphi_h) = & \sum_{K \in \mathscr{T}_h} \int_K \sum_{s=1}^{d} \mathbf{R}_s(\mathbf{w}_h, \nabla \mathbf{w}_h) \frac{\partial \varphi_h}{\partial x_s} \, \mathrm{d}x \\
& - \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \sum_{s=1}^{d} \langle \mathbf{R}_s(\mathbf{w}_h, \nabla \mathbf{w}_h) \rangle n_s [\varphi_h] \, \mathrm{d}S + \eta \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \sum_{s=1}^{d} \langle \mathbf{R}_s(\mathbf{w}_h, \nabla \varphi_h) \rangle n_s [\mathbf{w}_h] \, \mathrm{d}S \\
& - \sum_{K \in \mathscr{T}_h} \int_K \sum_{s=1}^{d} \mathbf{f}_s(\mathbf{w}_h) \frac{\partial \varphi_h}{\partial x_s} \, \mathrm{d}x + \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \mathbf{H}(\mathbf{w}_h|_\Gamma^{(L)}, \mathbf{w}_h|_\Gamma^{(R)}, \mathbf{n}_\Gamma) [\varphi_h]_\Gamma \, \mathrm{d}S \\
& + \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \sigma [\mathbf{w}_h] \, [\varphi_h] \, \mathrm{d}S
\end{aligned}$$

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$$+ \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \sigma [\mathbf{w}_h] \, [\varphi_h] \, \mathrm{d}S$$

finite volume method

DG discretization

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& + \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \sigma [\mathbf{w}_h] \, [\varphi_h] \,\mathrm{d}S
\end{aligned}$$

finite element method

DG discretization

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\end{aligned}$$

only DGM

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Numerical analysis

Asymptotical order of convergence

Poisson problem & scalar nonlinear convection-diffusion equation

$$|u - u_h|_{H^1(\Omega)} = O(h^p), \qquad \|u - u_h\|_{L^2(\Omega)} = O(h^{p+1})$$

the same order of convergence as for FEM

Number of degrees of freedom (DOF)

let $d = 2$, $M = \#\mathscr{T}_h$

p	1	2	3	4	5	6
FEM	$\frac{1}{2}M$	$2M$	$\frac{9}{2}$	$8M$	$\frac{25}{2}M$	$18M$
DGM	$3M$	$6M$	$10M$	$15M$	$21M$	$28M$
ratio= $\frac{\mathrm{DGM}}{\mathrm{FEM}}$	6	3	2.22	1.875	1.68	1.55

DGM is more expensive than FEM, the ratio is smaller for larger p

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Algebraic representation

Space-time discretization

$$\frac{1}{\tau_k}\left(\mathbf{w}_h^k - \mathbf{w}_h^{k-1}, \varphi_h\right) + \mathbf{c}_h(\mathbf{w}_h^k, \varphi_h) = 0 \quad \forall \varphi_h \in \mathbf{S}_{hp}, \ k = 1, \dots \quad (2)$$

- system of nonlinear algebraic equations for each k, let $\mathbf{w}_h^k \in \mathbf{S}_{hp} \leftrightarrow \xi_k \in \mathbb{R}^{\mathrm{DOF}}$, $\mathrm{DOF} = \dim \mathbf{S}_{hp}$, (2) is equivalent to

$$\frac{1}{\tau_k}\mathbb{M}\xi_k + \mathbf{F}(\xi_k) = \frac{1}{\tau_k}\mathbb{M}\xi_{k-1}, \ k = 1, 2, \dots$$

- Newton-like method uses $\mathbf{F}(\xi_k) \approx \mathbb{C}(\xi_k)\xi_k - \mathbf{b}(\xi_k)$,
- $\mathbb{M}$ and $\mathbb{C}$ sparse block-structure matrices

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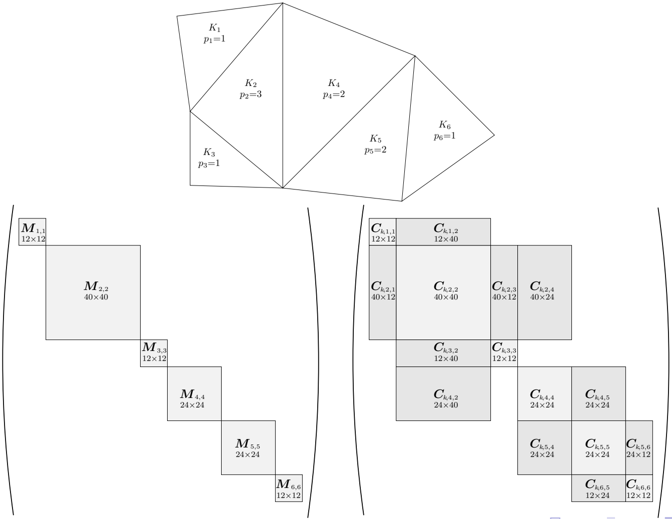
Block structure of $\mathbb{M}$ and $\mathbb{C}$



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- h-multigrid is based on hierarchy of grids $\mathscr{T}_h$, $h \in \{h_0, h_1, \dots\}$
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Solution of complex problems requires parallelization

- suitable discretization technique
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Parallelization of DGM

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- DGM allows to distribute efficiently the computation and required memory among the processors
- interchange of data among processors is minimal
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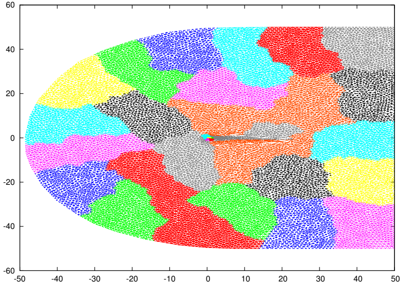
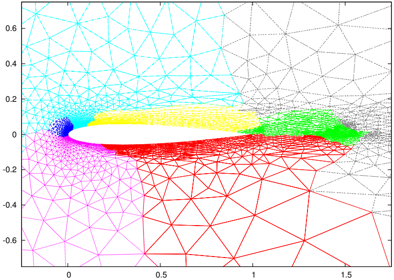
Mesh decomposition [V. Šíp, Master thesis 2011]





mesh decomposition among 32 processors

Efficiency of parallelization [V. Šíp, Master thesis 2011]

mesh	$\#\mathscr{T}$	P_1 DOF	P_3 DOF
$\mathscr{T}_{h1}$	6 876	82 512	275 040
$\mathscr{T}_{h2}$	29 040	348 480	1 161 600

π-number of processors, (1,2,4,8,16,32),
$S_\pi = T_1/T_\pi$ - acceleration,
$E_\pi = S_\pi/\pi$ - efficiency,
ideal case: $S_\pi = \pi$ & $E_\pi = 1$

		$\mathscr{T}_{h1}$			$\mathscr{T}_{h2}$		
	π	T_π	S_π	E_π	T_π	S_π	E_π
P_1	1	102.43 s	1.00	1.00	539.73 s	1.00	1.00
	2	50.75 s	2.02	1.01	258.16 s	2.09	1.05
	4	25.90 s	3.95	0.99	131.49 s	4.10	1.03
	8	15.07 s	6.80	0.85	65.80 s	8.20	1.03
	16	9.79 s	10.46	0.65	35.21 s	15.33	0.96
	32	7.77 s	13.19	0.41	26.75 s	20.18	0.63
P_3	1	1278.27 s	1.00	1.00	5988.53 s	1.00	1.00
	2	651.42 s	1.96	0.98	3026.76 s	1.98	0.99
	4	310.50 s	4.12	1.03	1526.50 s	3.92	0.98
	8	156.58 s	8.16	1.02	760.79 s	7.87	0.98
	16	87.28 s	14.65	0.92	395.66 s	15.13	0.94
	32	50.07 s	25.53	0.80	206.64 s	28.98	0.91

Efficiency of parallelization [V. Šíp, Master thesis 2011]

mesh	$\#\mathscr{T}$	P_1 DOF	P_3 DOF
$\mathscr{T}_{h1}$	6 876	82 512	275 040
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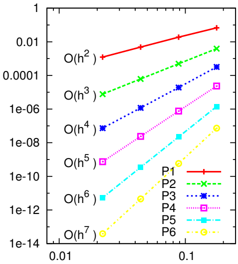
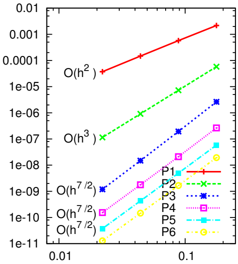
Orders of convergence

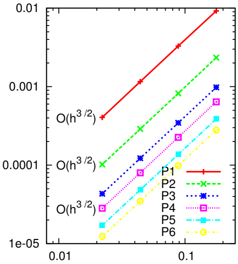
- higher order schemes are efficient
- BUT only if the exact solution is sufficiently regular
- $\|u - u_h\|_{L^2} = O(h^{\mu})$, $\mu = \min(p+1, s)$, $u_h \in S_{hp}$, $u \in H^s(\Omega)$



$u \in C^{\infty}(\Omega)$



$u \in H^{3.5}(\Omega)$



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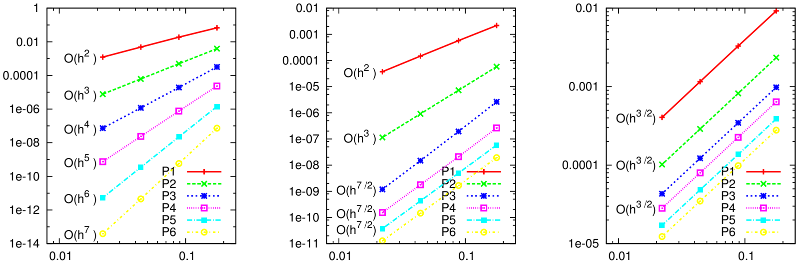

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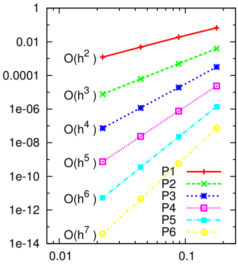

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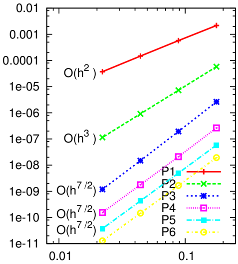

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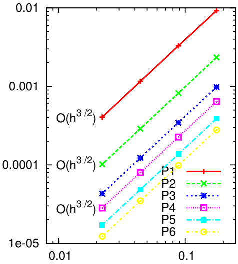

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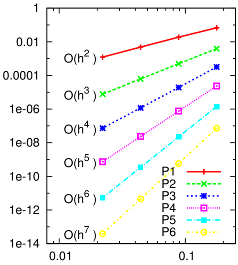

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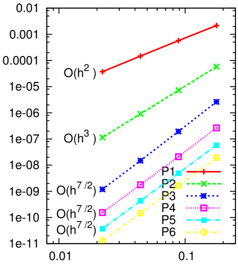

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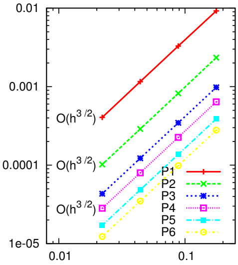

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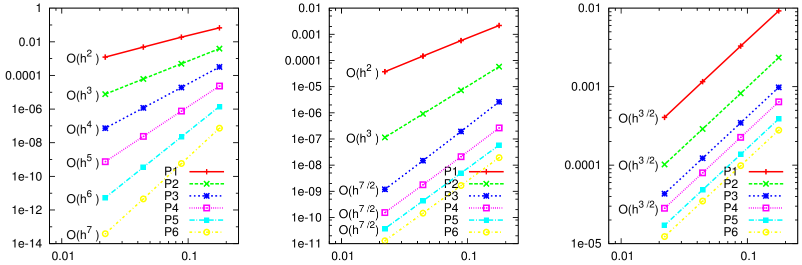

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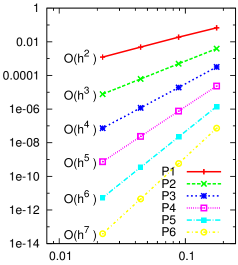

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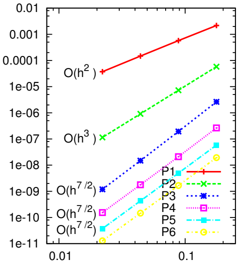

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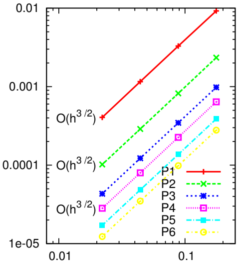

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hp-DGM

Idea of *hp*-methods

- *p*-refinement in subdomains where solution is regular,
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Technical aspects

- marking of elements for refinement (based on error estimates)
- decide if p or h adaptation
- many (more or less heuristic) techniques
  [Demkowicz, Georgoulis, Houston, Melenk, Schwab, Šolín, Süli, ...]

Implementation aspects of *hp*-DGM

- basis of S_h^p: local and discontinuous
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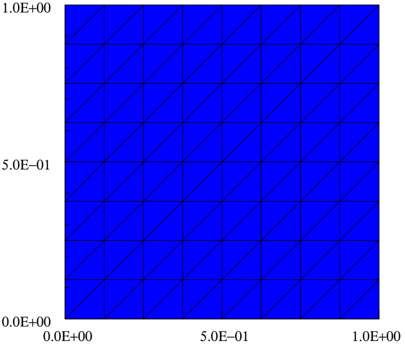
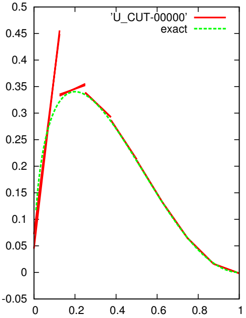
Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

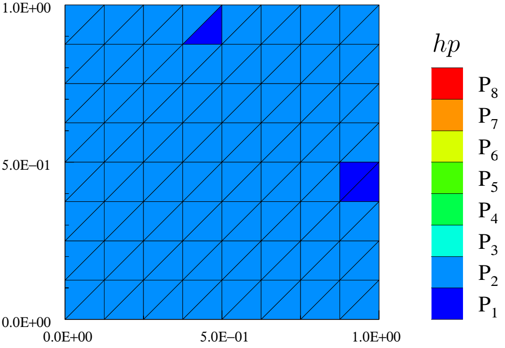


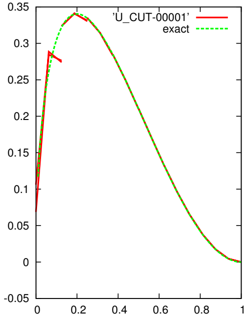


adaptation level = 00

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

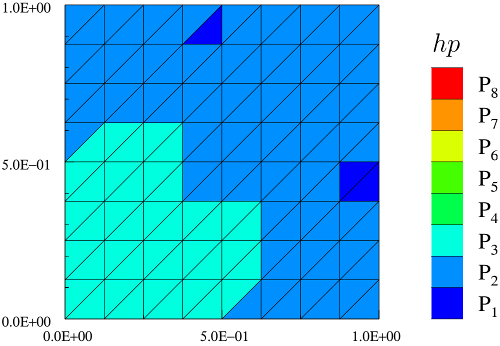


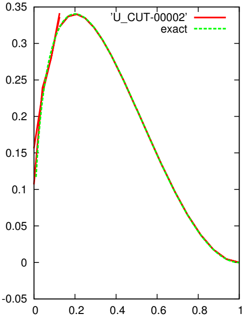


adaptation level = 01

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

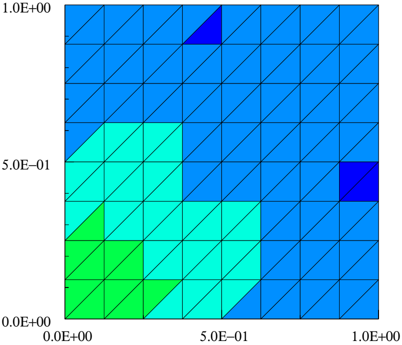


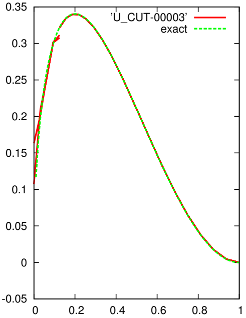


adaptation level = 02

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

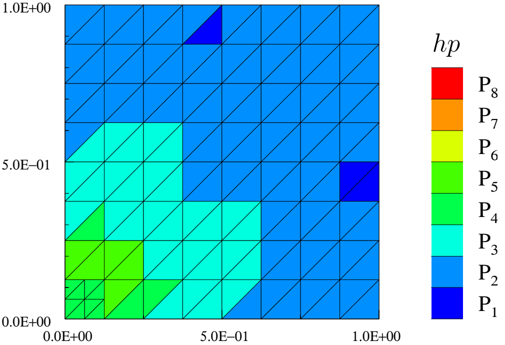


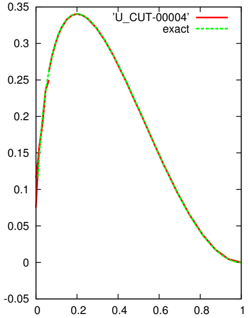


adaptation level = 03

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

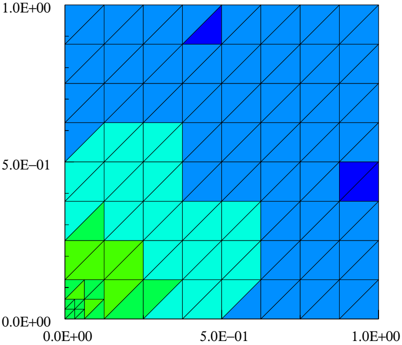


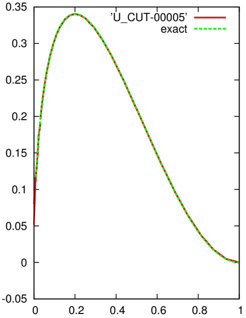


adaptation level $= 04$

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

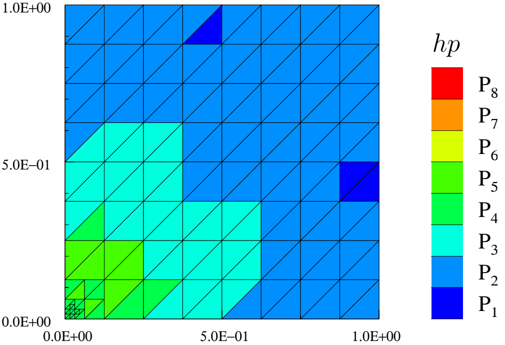


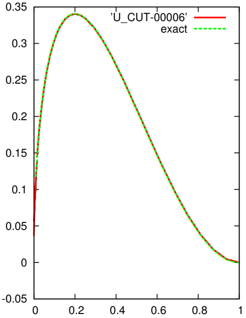


adaptation level = 05

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

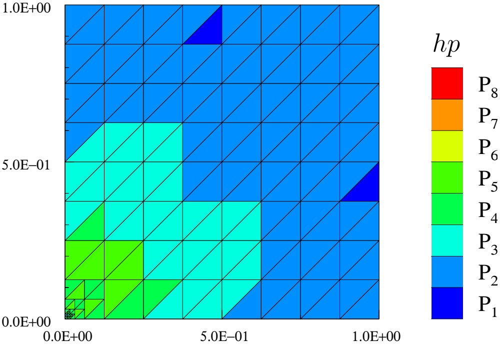


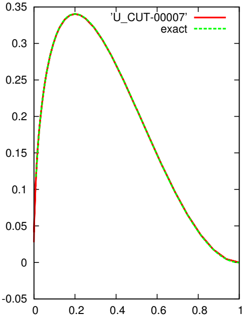


adaptation level = 06

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

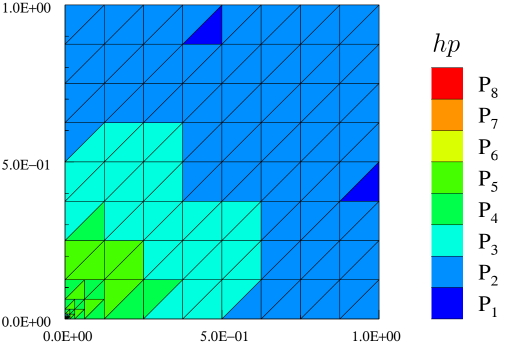


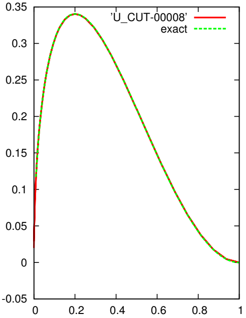


adaptation level = 07

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

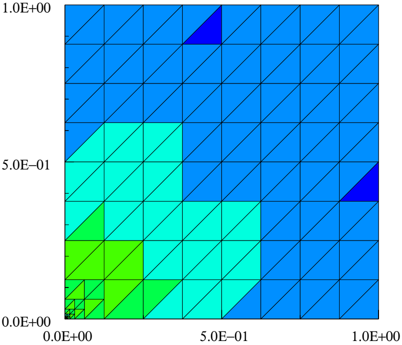


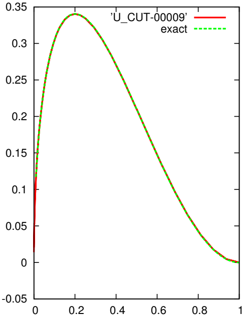


adaptation level = 08

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

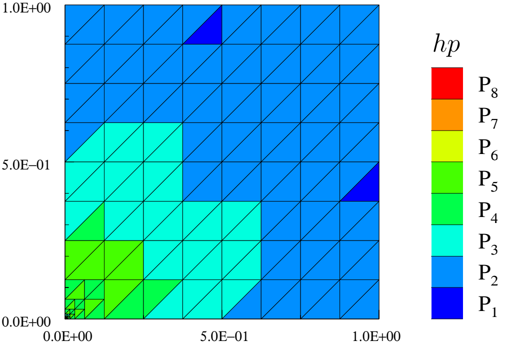


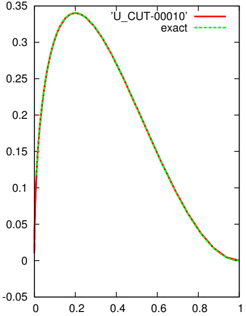


adaptation level = 09

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity

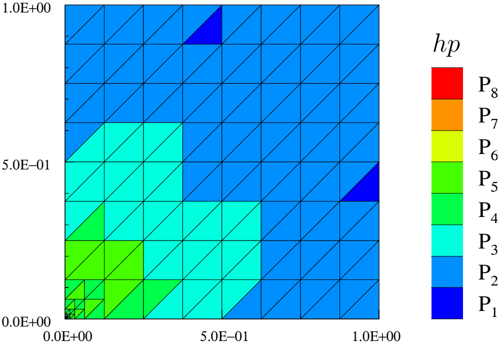


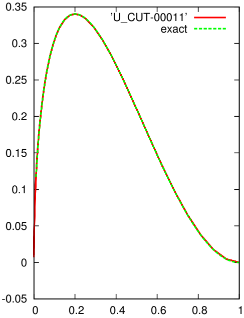


adaptation level = 10

Performance of the *hp*-DGM

nonlinear convection-diffusion equation with a corner singularity





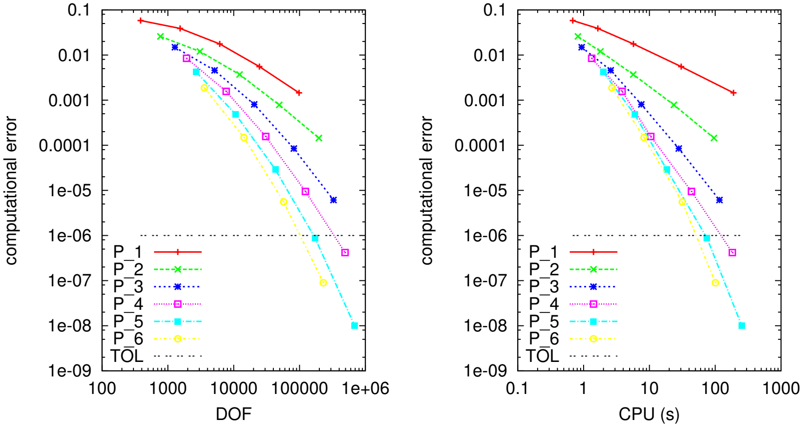
adaptation level $= 11$

Practical demonstration of the efficiency of *hp*-DGM

- scalar linear convection-diffusion equation solved by DGM

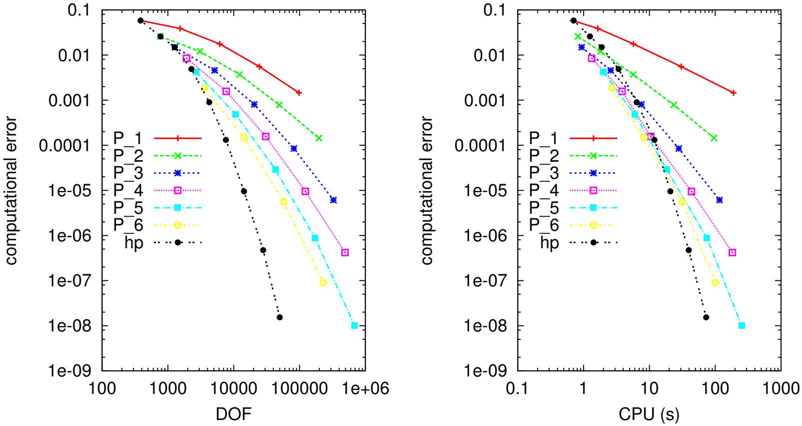
Practical demonstration of the efficiency of *hp*-DGM

- scalar linear convection-diffusion equation solved by DGM



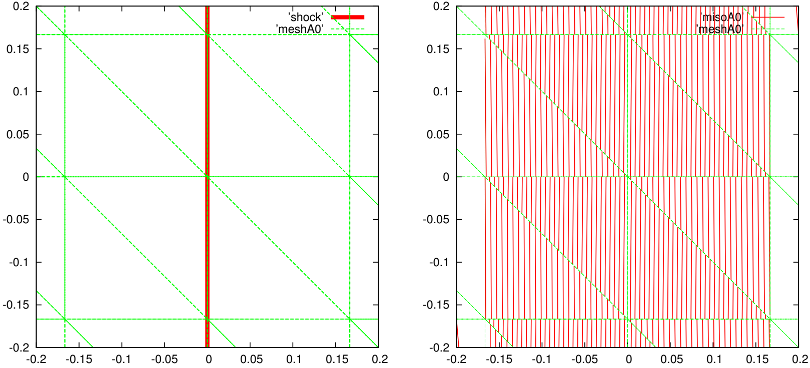
Practical demonstration of the efficiency of *hp*-DGM

- scalar linear convection-diffusion equation solved by DGM



h-refinement for interior and boundary layers problems

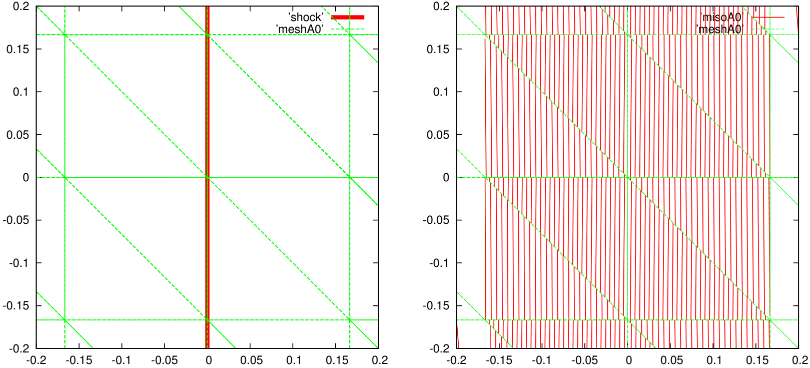
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient



adaptation level = 0

h-refinement for interior and boundary layers problems

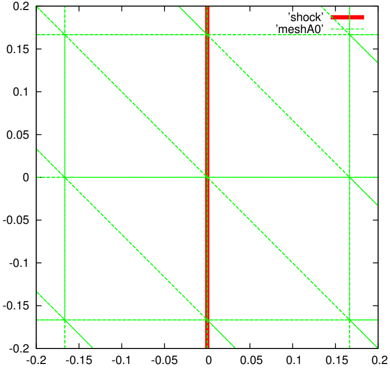
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

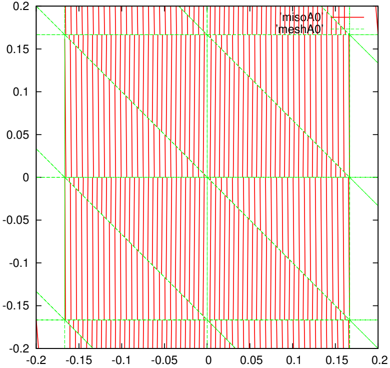


adaptation level $= 0$

h-refinement for interior and boundary layers problems

- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

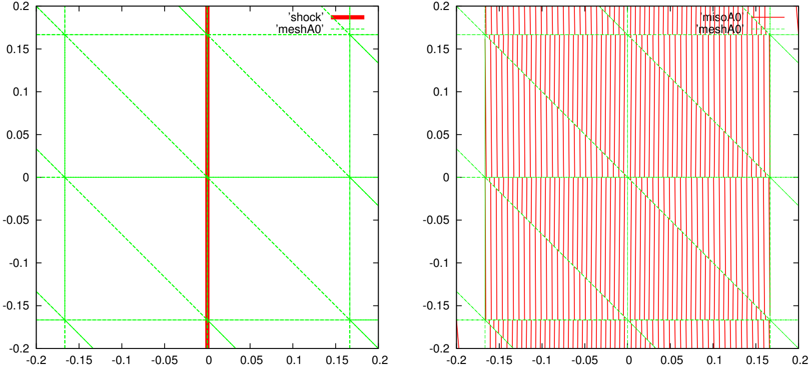




adaptation level $= 0$

h-refinement for interior and boundary layers problems

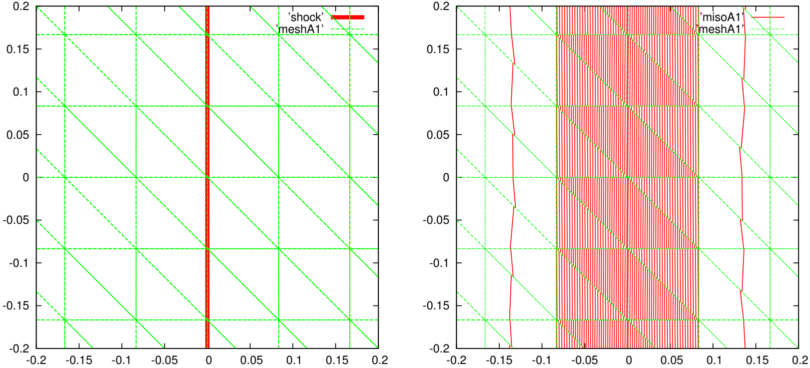
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient



adaptation level $= 0$

h-refinement for interior and boundary layers problems

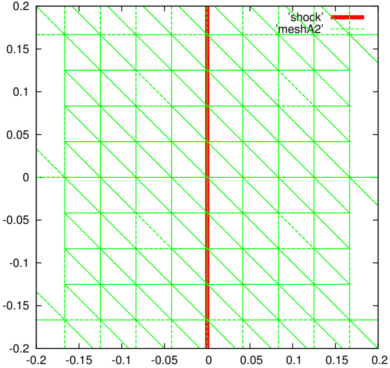
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

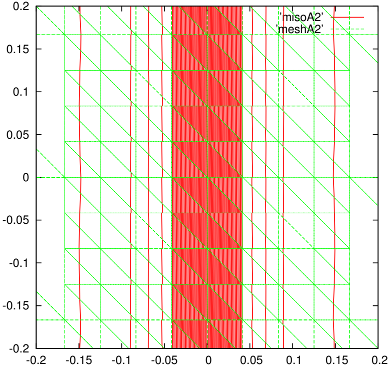


adaptation level $= 1$

h-refinement for interior and boundary layers problems

- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

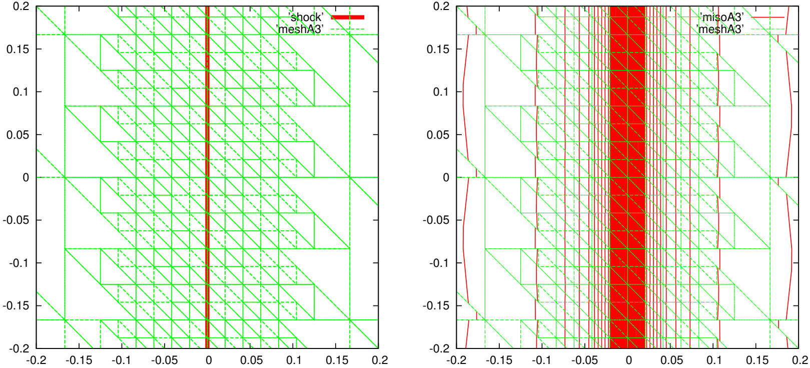




adaptation level = 2

h-refinement for interior and boundary layers problems

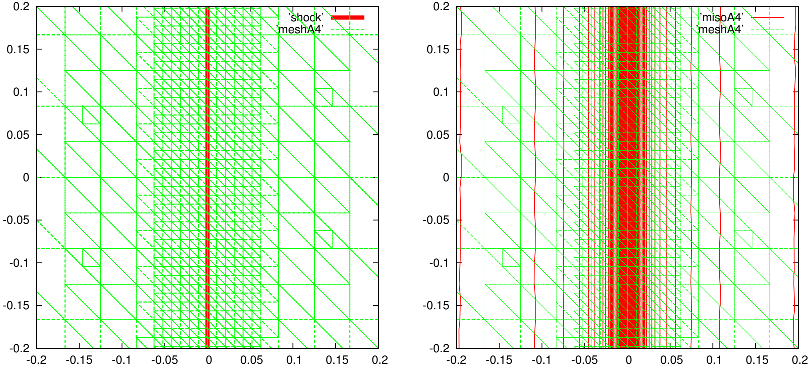
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient



adaptation level = 3

h-refinement for interior and boundary layers problems

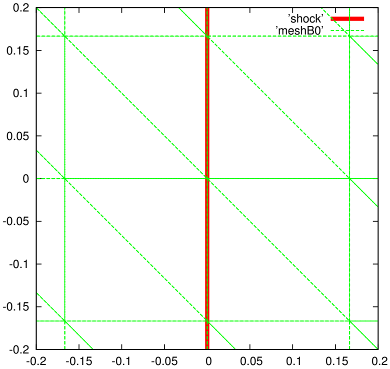
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

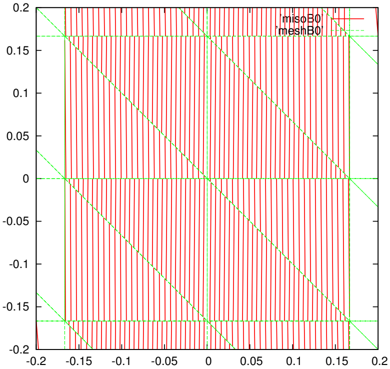


adaptation level = 4

h-refinement for interior and boundary layers problems

- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

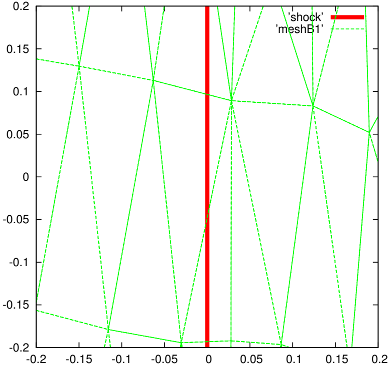


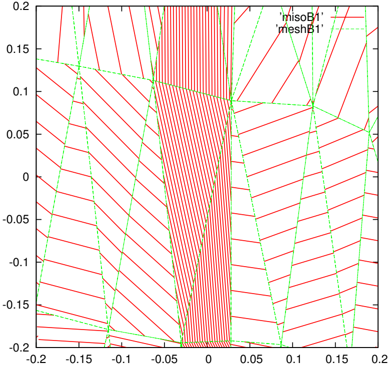


AMA adaptation level = 0

h-refinement for interior and boundary layers problems

- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

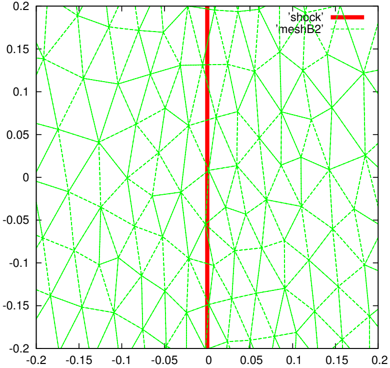


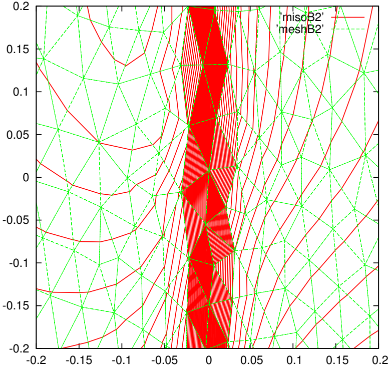


AMA adaptation level = 1

h-refinement for interior and boundary layers problems

- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

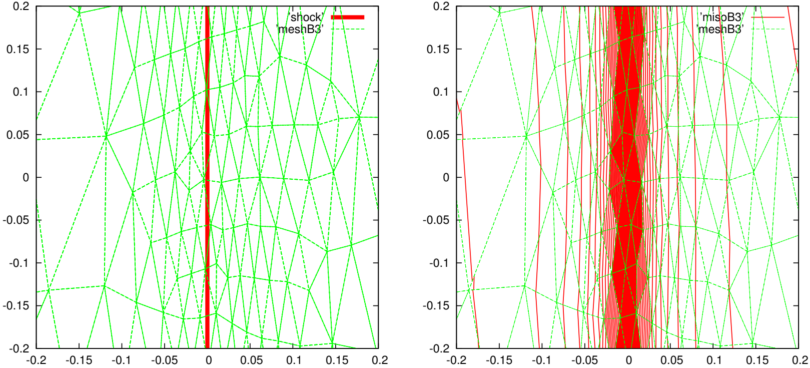




AMA adaptation level = 2

h-refinement for interior and boundary layers problems

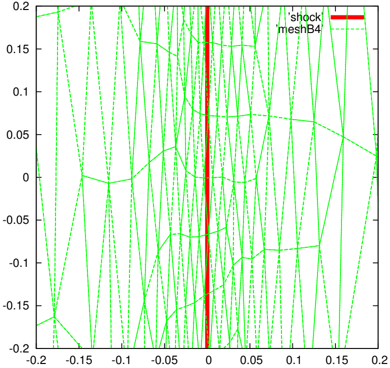
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

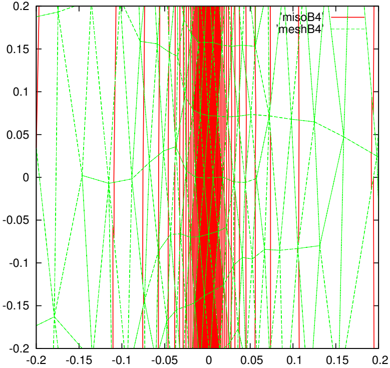


AMA adaptation level = 3

h-refinement for interior and boundary layers problems

- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

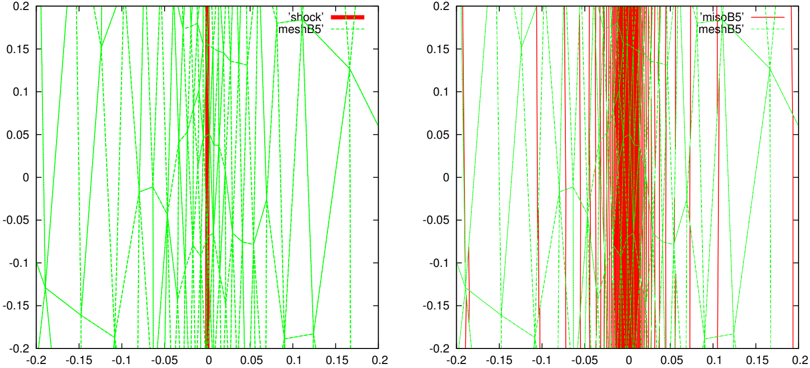




AMA adaptation level = 4

h-refinement for interior and boundary layers problems

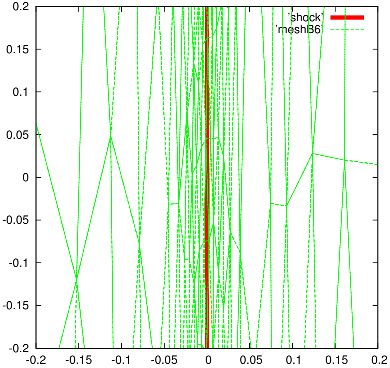
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

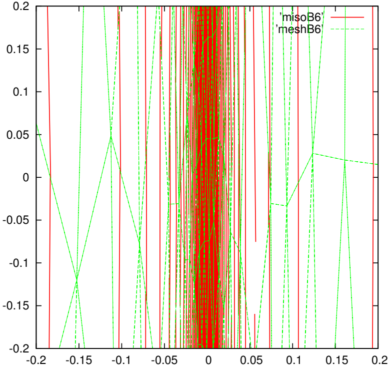


AMA adaptation level = 5

h-refinement for interior and boundary layers problems

- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient

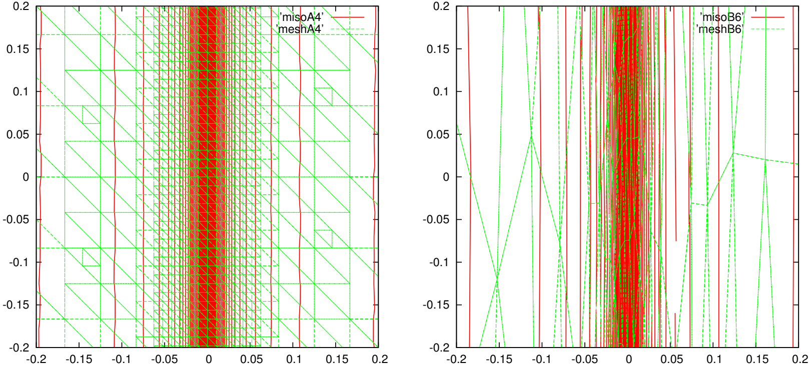




AMA adaptation level = 6

h-refinement for interior and boundary layers problems

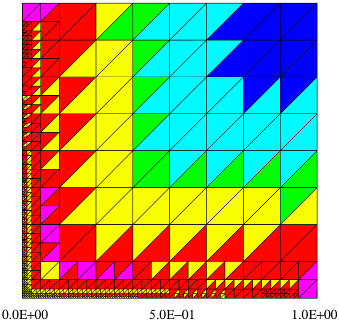
- isotropic *h*-refinement – not efficient
- anisotropic mesh adaptation – more efficient



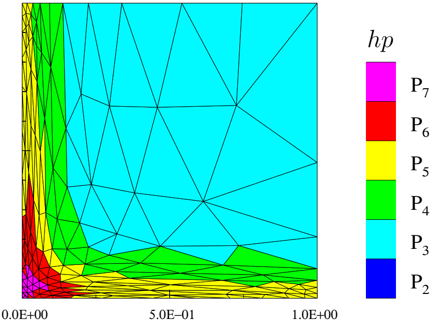
comparison: isotropic (5649 elems) – anisotropic (432 elems)

Anisotropic *hp*-DGM

hp-DGM & anisotropic mesh adaptation



isotropic *hp*-mesh



anisotropic *hp*-mesh

Anisotropic *hp*-DGM

Basic idea [D. APNUM 2014]

- let $u : \Omega \to \mathbb{R}$, $q > 1$ and $\omega > 0$ be given,
- we seek mesh $\mathscr{T}_h$ and the corresponding space S_{hp} such that
  - $\|u - \Pi_{hp} u\|_{L^q(\Omega)} \leq \omega$,
  - $\dim S_{hp}$ is minimal,
- Π_{hp} is the L^2-orthogonal projection into S_{hp}.

Practical realization

- for each $x \in \Omega$ and $p \in \mathbb{N}$, we seek optimal shape of $K_{x,p}$ minimizing $\|u - \Pi_{hp} u\|_{L^q(K)}$,
- from $\|u - \Pi_{hp} u\|^q_{L^q(K)} \leq \omega^q \frac{|K|}{|\Omega|}$, we set size of $K_{x,p}$
- we set optimal p for $x \implies$ "optimal" S_{hp}.

Anisotropic *hp*-DGM

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- from $\|u - \Pi_{hp} u\|^q_{L^q(K)} \leq \omega^q \frac{|K|}{|\Omega|}$, we set size of $K_{x,p}$
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Anisotropic *hp*-DGM

Basic idea [D. APNUM 2014]

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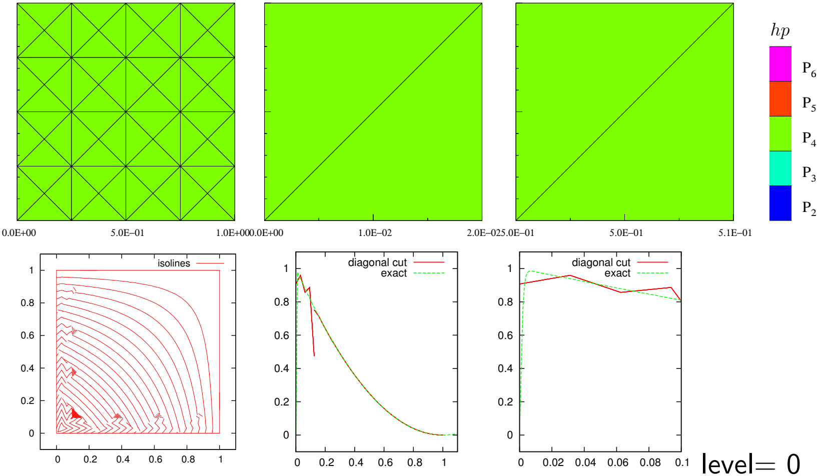
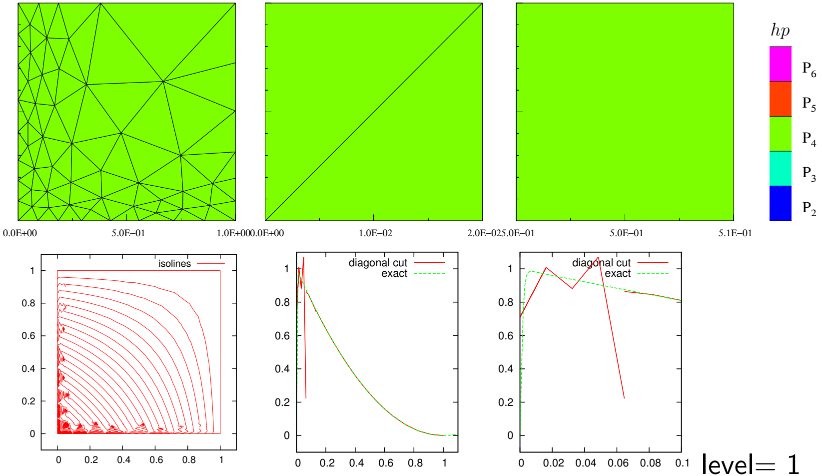
Boundary layers with $\varepsilon = 10^{-3}$ – performance



level= 0

Boundary layers with $\varepsilon = 10^{-3}$ – performance



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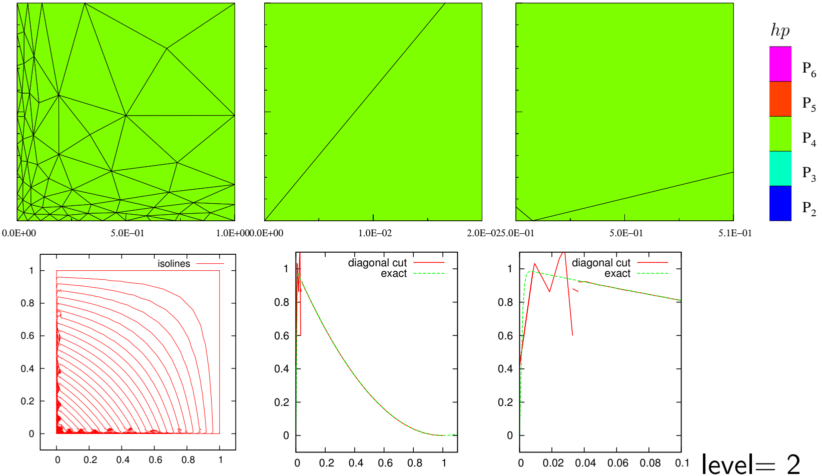
Boundary layers with $\varepsilon = 10^{-3}$ – performance



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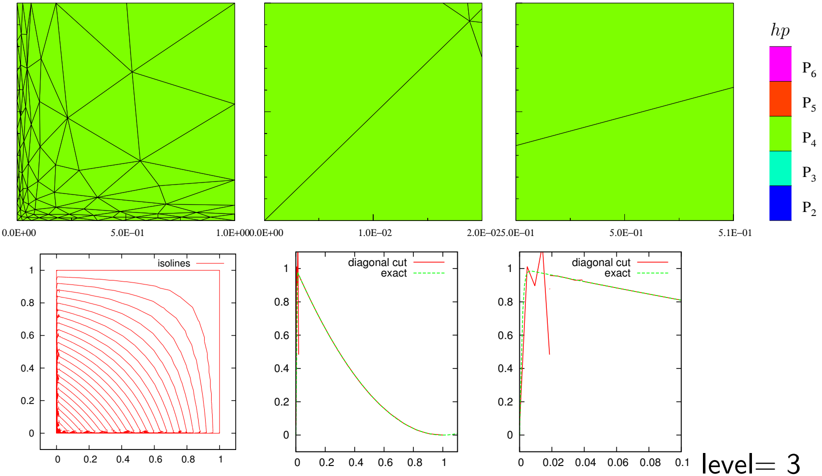
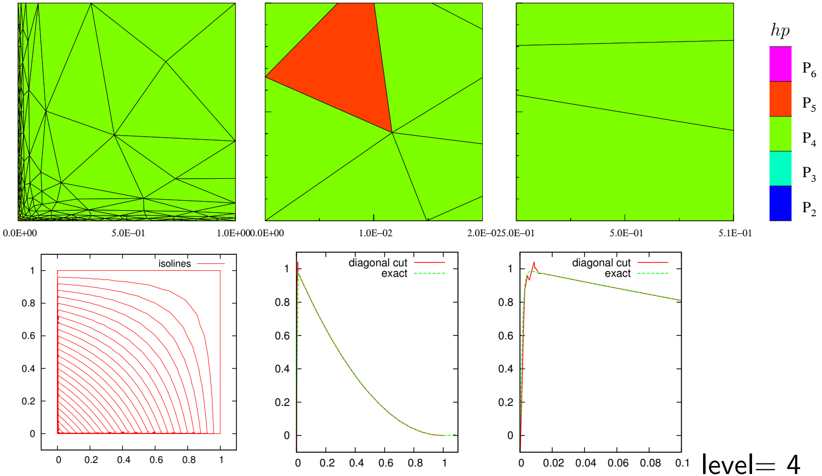
Boundary layers with $\varepsilon = 10^{-3}$ – performance



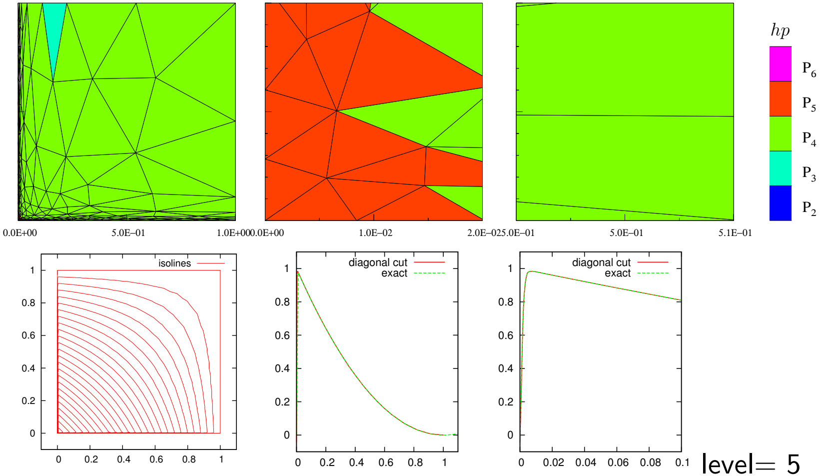
level$= 3$

Boundary layers with $\varepsilon = 10^{-3}$ – performance



level= 4

Boundary layers with $\varepsilon = 10^{-3}$ – performance



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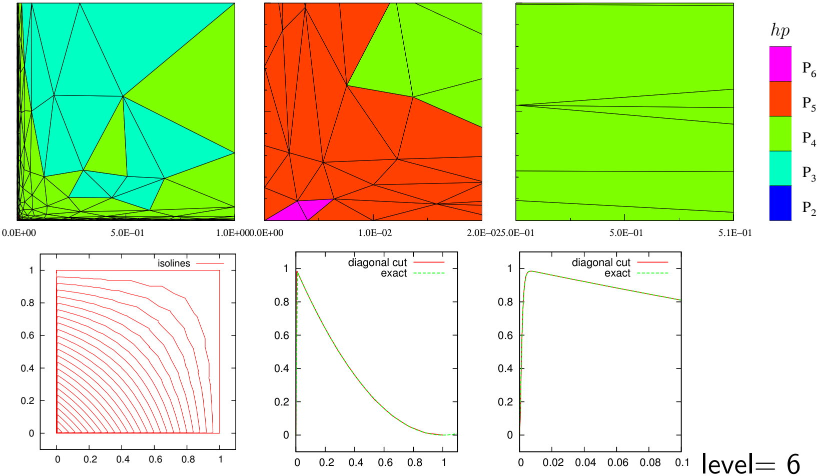
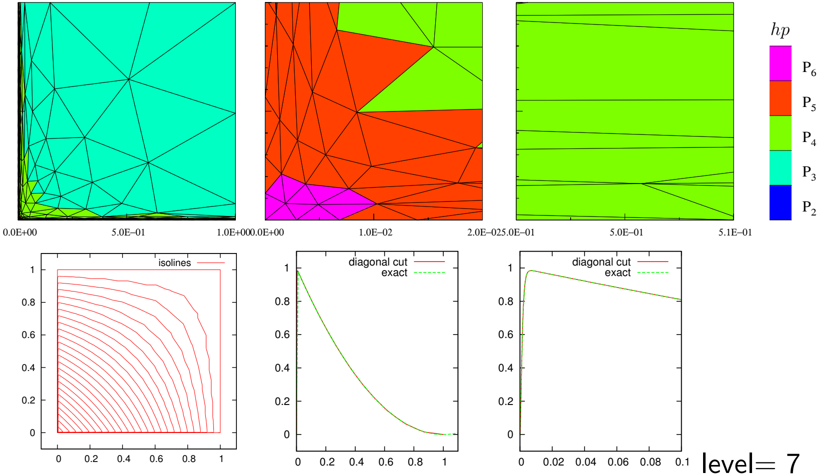
Boundary layers with $\varepsilon = 10^{-3}$ – performance



level$= 6$

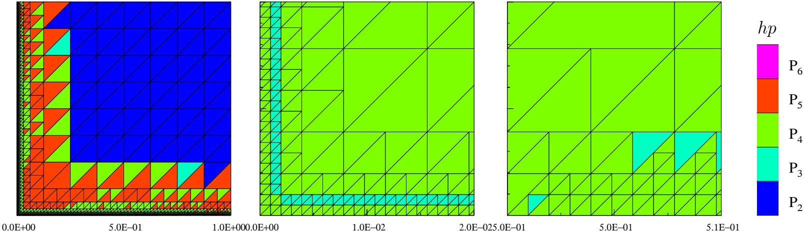
Boundary layers with $\varepsilon = 10^{-3}$ – performance



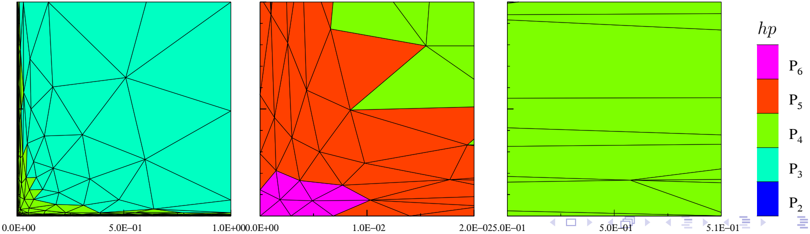
level= 7

Boundary layers with $\varepsilon = 10^{-3}$ – comparison

isotropic: total — zoom 50 of origin — zoom 100 of boundary layer



anisotropic



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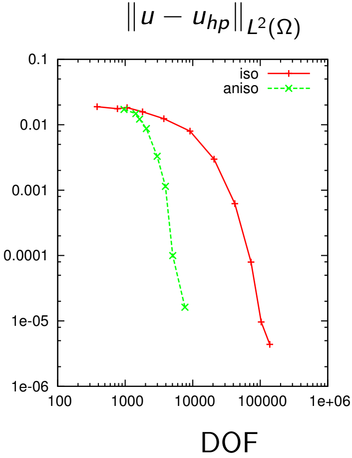

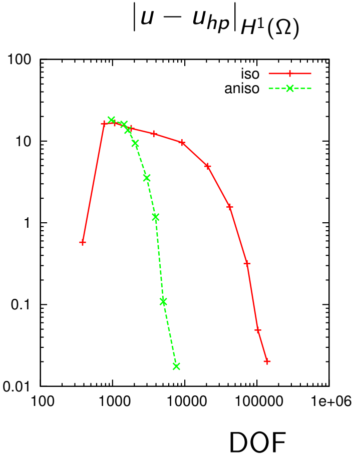

Anisotropic *hp*-DGM for compressible flow simulation

Viscous compressible flow around NACA 0012

$M_{\rm inlet} = 0.5$, $\alpha = 3^\circ$, $\mathrm{Re} = 5\,000$,

Rerefence values of aerodynamical coefficients

$c_D = 0.057701, \quad c_L = 0.052737, \quad c_M = -0.022987$

The goal: compute aerodynamical coefficients

with tolerances: 1% for c_D, 5% for c_L, 2% for c_M

Adaptive techniques

- isotropic *hp*-refinement
- P_1 and P_2 anisotropic mesh refinement
- anisotropic *hp*-refinement

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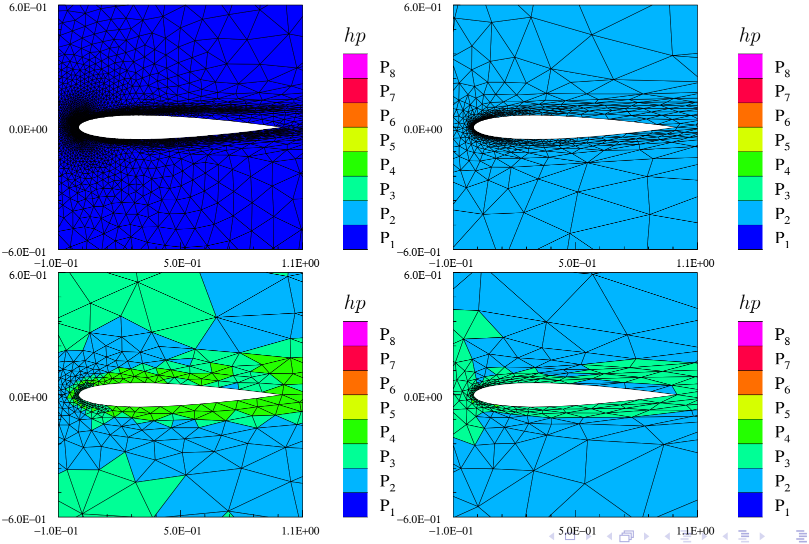
Comparison of adaptations: *hp*-grids



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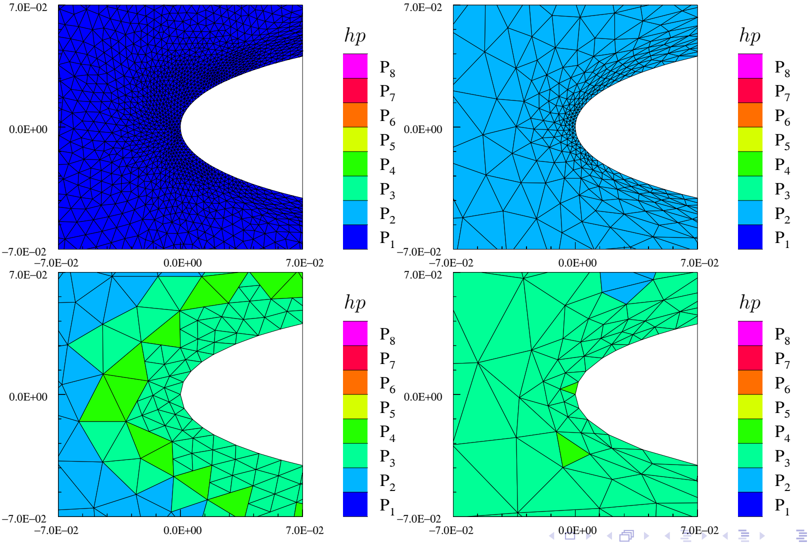

Comparison of DOF and CPU

method	$\#\mathscr{T}_h$	DOF	CPU(s)
AMA P_1	7968	95616	266
AMA P_2	1618	38832	107
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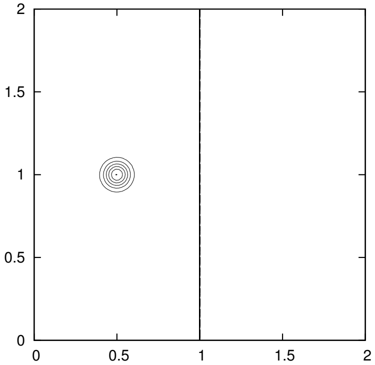
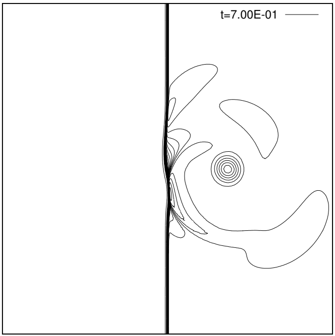
Viscous shock-vortex interaction - preliminary results





Viscous shock-vortex interaction – adaptive refinement

A posteriori error estimates

State of the art

- many theoretical works for scalar linear problems,
- nonlinear problems: significantly smaller number of results
- most results estimates of $\|\mathbf{w} - \mathbf{w}_h\|$ in a suitable norm
- It is really our goal?

A posteriori error estimates of the quantity of interest

- quantity of interest: a linear functional $J(\mathbf{w})$,
- estimate $J(\mathbf{w}) - J(\mathbf{w}_h)$
- theory [Becker, Rannacher, Suli, Giles, Hartmann, ...],
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